Upsala J Med Sci 83: 89–95, 1978

The Inherent Mathematical Error Factors in the Radiological Determination of Heart Volume

G. O. SPERBER

From the Department of Physiology and Medical Biophysics, Uppsala, Sweden

ABSTRACT

When heart volume is calculated from two X-ray projections, part of the error is due to the fact that the formulas employed are not mathematically exact. This error was calculated for a variety of shapes and positions of the heart. The results indicate that this error may reach about 30 % for hearts of fairly ordinary shape. When the standard projections are used, horizontally positioned hearts are particularly liable to give rise to large errors of this kind.

INTRODUCTION

In the routine determination of heart volume from chest X-ray films, the heart is assumed to have the shape of an ellipsoid, and the volume of this ellipsoid is calculated by the multiplication of three distances ("axes") measured from the films, and a constant factor. The most important source of error in this procedure probably lies in the estimation of the position of parts of the heart border, but there are several more.

The actual volume of the heart may be expressed as

$$V=\frac{\pi}{6}\cdot e_1\cdot l_1\cdot e_2\cdot l_2\cdot e_3\cdot l_3\cdot \theta_1^2\cdot \theta_2\cdot S\cdot G \qquad (1)$$

where e_1, e_2, e_3 are the errors in the estimates of the distances to be measured whose actual values are l_1, l_2, l_3. θ_1 and θ_2 are enlargement factors, which would be unity if the films could be put in the "ideal film plane", i.e. the plane passing through the centre of the heart perpendicularly to the central X-ray, but in practice they will be larger because of the divergence of the rays, and S is a shape factor because the heart is not really ellipsoidal. G, finally, is a geometrical factor because the distances l_1, l_2, l_3 do not generally coincide with the main axes of the ellipsoid so that the formula $\pi/6\cdot l_1\cdot l_2\cdot l_3$ for the volume is not mathematically valid.

In routine practice, average values of e_1, e_2, e_3, θ_1, θ_2, G and S are usually grouped with $\pi/6$ into a single empirical constant. When more individualized values are needed, these can easily be obtained for θ_1 and θ_2 from measurements of the focus-film and heart-film distances. S is more difficult to handle, but some aid can be had from the work on heart models by Bergström (1). G is dependent on the shape and orientation of the heart and can be calculated mathematically, which is the subject of this paper.

No such calculations of G seem to have been published, it being usually tacitly assumed that the variations in G are negligible in practice. This seems a reasonable assumption, in view of the many other sources of error, but it is not self-evident. Also, at the time when the methods were originally being developed, the required amount of computing effort must have seemed forbidding, but it can easily be handled with computer.

Therefore, this paper presents calculations of G for various shapes and positions of the heart ellipsoid, aiming to determine the variations in calculated volume introduced by this factor.

In these calculations, account is taken of two different ways of defining l_1, l_2 and l_3. According to Jonsell (2) (the commonly used method) l_1 and l_2 are the lengths of the main axes of the heart shadow on the frontal film, and l_3 is the longest sagittal chord of the heart shadow on the lateral film (Fig. 1). Alternatively, according to Kjellberg et al. (4) l_1 and l_2 may be taken as the overall width and overall height of the heart shadow on the frontal film. These authors also recommended a 30° angulation of the tube, and an accompanying correction factor in the formula ($\sqrt{1-l_3/3l_2}$, corrected for magnification), but this has not been included in this work, as the angulation of the tube has not found favour in practice.

METHODS

The mathematical model indicated in Fig. 2 was used. It assumes that the X-ray source is a point and that the central ray passes through the centre of the heart.

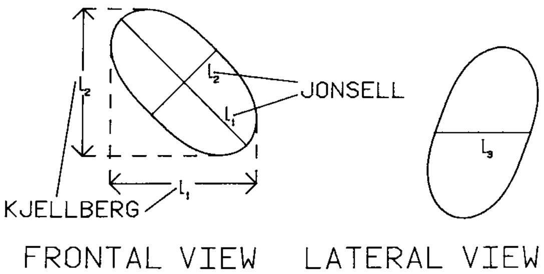

Fig. 1. The theoretical parameters l_1, l_2 and l_3 of the formulas of Jonsell and Kjellberg.

An ellipsoid with half-axes a, b, c ($a \geq b \geq c$) has its centre at the origin of a three-dimensional orthogonal coordinate system. At distance U along the x-axis, there is one ray source, which projects the ellipsoid onto the "ideal film plane", the yz-plane (equation $x=0$). The lateral projection is from a ray source at distance V along the y-axis onto the xz-plane ($y=0$).

These projections are ellipses, whose equations were determined as described in Appendix I. From these equations were calculated the frontal measurement distances l_1 and l_2 as defined by Jonsell (Appendix II*a*) and Kjellberg (Appendix II*b*) and l_3, the distance on the lateral projection common to these methods (Appendix II*c*). The product of these axes was divided by $8 \cdot a \cdot b \cdot c$ to yield G.

The calculations were programmed in Nord Compiler Basic and performed on a Nord-10 computer.

RESULTS

121 different ellipsoid shapes were investigated, with the axis ratios a/b and b/c varying between 1 and 2. The absolute parameter values, which are not of critical importance in this connection, were chosen: $b=4$ cm, $U=140$ cm, $V=130$ cm. For each ellipsoid, G was calculated according to Jonsell's and Kjellberg's methods in 1 170 different positions, viz. with the long axis in 65 different directions spread throughout one quadrant of space, and with 18 rotations (spaced at 10° intervals) around the long axis in each of the 65 directions. The positions giving maximum and minimum G were noted for each shape.

The minimum possible G invariably occurred when the axes of the ellipsoid were closely parallel to the coordinate axes, and it was in all cases 1.001 according to both methods.

The maximum possible G was also found to have the same value for both methods, but varied according to the shape of the ellipsoid, as shown in Table I. It generally occurred when the second longest axis of the ellipsoid was parallel to the z coordinate axis, and the long axis was in a direction midway between the x and y coordinate axes.

Another parameter, the "sagittal diameter quotient" was also calculated. It was defined as the sagittal diameter of the lateral heart projection (i.e. l_3) divided by the actual sagittal (i.e. in the direction of the x-axis) diameter of the ellipsoid, determined as described in Appendix III. It was found to coincide with G to two decimal places, but was up to three units smaller in the third decimal position.

Also, the variations in G when one specific ellipsoid (axis ratio 1.6 : 1 : 0.75) rotated about its vertical (i.e. in the direction of the z-axis) second-longest axis were investigated. This was done in order to illustrate the high sensitivity of G to rotation in horizontal (i.e., with the long axis in the transversal plane of the body) hearts. The results are shown in Fig. 3.

DISCUSSION

From the results, it is seen how the geometrical error factor G is caused. The whole method is founded on the formula of Kahlstorf (3) which states that the volume of any ellipsoid is 2/3 of the product of the area of the frontal projection of the ellipsoid and the sagittal diameter of the ellipsoid. By methods similar to those described in the Appendices this formula can be shown to be mathematically exact (Kahlstorf's proof, however, is not complete), but when it is applied in the radiological context, two error factors occur. The important one of these is expressed in the "sagittal diameter quotient": the sagittal diameter of the *heart* cannot

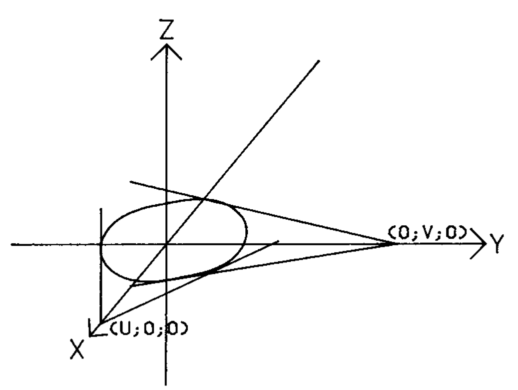

Fig. 2. The coordinate system used for the calculations, with an ellipsoid centered at the origin and ray sources at (U; 0; 0) and (0; V; 0).

Table I. *The maximum values of the geometrical error factor G, obtainable by rotations of 121 ellipsoids with one axis = 8 cm and the other axes varying as shown*

The "sagittal diameter quotient", calculated under the same premises, also conforms with this table to two decimal places

Long axis, 2*a*, cm	Short axis, 2*c*, cm 4.0	4.4	4.8	5.2	5.6	6.0	6.4	6.8	7.2	7.8	8.0
8.0	1.25	1.18	1.13	1.10	1.06	1.04	1.03	1.01	1.01	1.0	1.0
8.8	1.33	1.25	1.19	1.14	1.10	1.08	1.05	1.04	1.03	1.01	1.01
9.6	1.41	1.32	1.25	1.19	1.15	1.11	1.09	1.06	1.04	1.03	1.02
10.4	1.50	1.40	1.32	1.25	1.20	1.16	1.12	1.09	1.07	1.05	1.04
11.2	1.58	1.47	1.38	1.31	1.25	1.20	1.16	1.13	1.10	1.08	1.06
12.0	1.67	1.55	1.45	1.37	1.31	1.25	1.21	1.17	1.14	1.11	1.09
12.8	1.76	1.63	1.52	1.44	1.36	1.30	1.25	1.21	1.17	1.14	1.12
13.6	1.85	1.71	1.60	1.50	1.42	1.36	1.30	1.25	1.21	1.18	1.15
14.4	1.94	1.80	1.67	1.57	1.48	1.41	1.35	1.30	1.25	1.22	1.18
15.2	2.04	1.88	1.75	1.64	1.55	1.47	1.40	1.35	1.30	1.26	1.22
16.0	2.13	1.96	1.82	1.71	1.61	1.53	1.46	1.40	1.34	1.30	1.26

be seen, and one will have to be content with the sagittal diameter of the heart's lateral *projection*. A second error is caused by the divergence of the rays, even after correction for magnification, but if the ray source is reasonably distant, this error has no practical importance. In the circumstances analyzed here, it did not exceed 0.3 %.

It is now easy to understand why *G* is largest for horizontal hearts in intermediate rotation; in this position, the measured sagittal diameter is determined mainly by the long axis of the heart and has hardly anything to do with the actual sagittal diameter.

Some of the values of *G* in Table I may seem appallingly distant from unity, but it must be remembered that values resulting from shapes and positions that do not occur in practice are not only practically but also theoretically without interest. This is because any method that attempts to estimate the volume of the heart (or of any other object) from only two projections must necessarily assume that the shape of the heart keeps within some kind of bounds. This necessity is illustrated, as pointed out by Bergström (1), by the fact that any pair of heart projections could in principle be caused by a suitably positioned disc with zero thickness.

The limits for the shapes included in Table I are rather wide. For example, the value in the lower left corner corresponds to a flat horizontal heart in intermediate rotation, twice as long as wide. While such a heart is not entirely impossible, it is certainly strange-looking.

However, the table indicates that *G* may vary considerably, even for more normal shapes. E.g. if a heart with the axis ratios as 1.6 : 1 : 0.75 (marked in Table I) is regarded as the limit for a reasonable shape, the table shows that a horizontal heart in intermediate rotation may be calculated to have 30 % larger volume than the very same heart if vertical or rotated counter-clockwise. This is hardly a negligible error.

Bergström (1) investigated a model ellipsoid experimentally. He found, with Jonsell's method, about 20 % variability. Since the axis lengths of the model were approximately 1.27 : 1 : 0.69, this agrees very well with the results shown in Table I. However, Bergström obtained primarily negative deviations from the ideal. The reason for this has to be that in that work, a strictly practical view was taken, and the measurements likely to be made in practice were used, rather than the ideal distances,

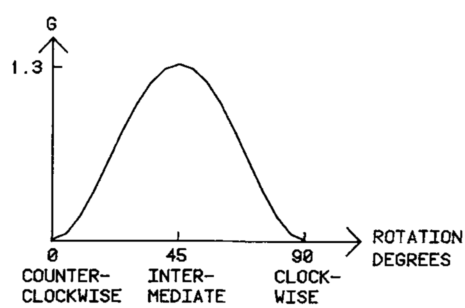

Fig. 3. The influence on *G* of rotation in the transversal plane of an ellipsoid with axis ratios 1.6 : 1 : 0.75, horizontally positioned.

so that the error factors e_1, e_2, e_3 came to play a role.

The results do not suggest any improvement in the formulas, since the problem is insufficient information. They do suggest that the radiologist should be sceptical against the routinely computed heart volume in horizontal hearts, where the mathematical error *may* be considerable, but is highly dependent on the rotation in the transversal plane (Fig. 3). In this case, it may be advisable to use additional projections, especially if high accuracy is required, or if the heart makes an impression of being elongated.

APPENDIX I

Calculation of the equations for the projections of the heart ellipsoid onto the "ideal film planes"

First, a coordinate system x_0, y_0, z_0 is laid with its axes coinciding with the axes of the ellipsoid, so that its equation becomes

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}=1 \qquad (2)$$

The ray sources will be situated at points $(A; B; C)$ (where $\sqrt{A^2+B^2+C^2}=U$) and $(D; E; F)$ (where $\sqrt{D^2+E^2+F^2}=V$). Since they are in orthogonal directions from the origin, $AD+BE+CF=0$.

Any ray through $(A; B; C)$ satisfies the equations

$$x_0=A+\alpha t$$
$$y_0=B+\beta t$$
$$z_0=C+\gamma t \qquad (3)$$

where α, β, γ depend on the direction of the ray. It cuts the ellipsoid at the points satisfying the equation obtained by eliminating x_0, y_0, z_0 between (2) and (3). After rearrangement this equation is

$$t^2\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)+2t\left(\frac{A\alpha}{a^2}+\frac{B\beta}{b^2}+\frac{C\gamma}{c^2}\right)+\left(\frac{A^2}{a^2}+\frac{B^2}{b^2}+\frac{C^2}{c^2}-1\right)=0 \qquad (4)$$

If this equation has no real roots, this means that the ray passes outside the ellipsoid. If there are two distinct roots it passes through the ellipsoid, and if there is one double root, it is a tangential ray. The latter happens if the discriminant of (4) is zero, i.e.

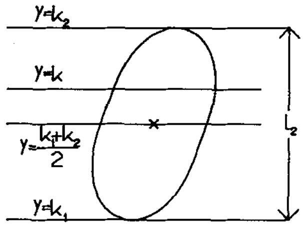

Fig. 4. See text, Appendix II.

$$\left(\frac{A\alpha}{a^2}+\frac{B\beta}{b^2}+\frac{C\gamma}{c^2}\right)^2-\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)\times\left(\frac{A^2}{a^2}+\frac{B^2}{b^2}+\frac{C^2}{c^2}-1\right)=0 \qquad (5)$$

Thus, if α, β, γ satisfy this equation, then (3) defines a tangential ray to the ellipsoid.

If α, β and γ are eliminated between (3) and (5) and terms are rearranged, the following equation results:

$$\left(\frac{Ay_0-Bx_0}{ab}\right)^2+\left(\frac{Bz_0-Cy_0}{bc}\right)^2+\left(\frac{Cx_0-Ay_0}{ac}\right)^2=\left(\frac{x_0-A}{a}\right)^2+\left(\frac{y_0-A}{b}\right)^2+\left(\frac{z_0-C}{c}\right)^2 \qquad (6)$$

This, then, is the equation of any point $(x_0; y_0; z_0)$ on the conical surface with apex at $(A; B; C)$ and tangential to the ellipsoid.

The corresponding conical surface with apex at $(D; E; F)$ will have a similar equation (7) with D, E and F substituted for A, B and C.

If now the Euclidean[1] transformation

$$x_0=x\frac{A}{U}+y\frac{D}{V}+z\frac{G}{W}$$

$$y_0=x\frac{B}{U}+y\frac{E}{V}+z\frac{H}{W}$$

$$z_0=x\frac{C}{U}+y\frac{F}{V}+z\frac{I}{W}$$

where $G=CE-FB$, $H=FA-CD$, $I=BD-AE$, $W=\sqrt{G^2+H^2+I^2}$ is applied, the system is rotated into the x, y, z-system defined previously, and (6) is transformed into

[1] A Euclidean transformation rotates space without affecting distances or angles. For reading on this matter the reader is referred to a textbook on linear algebra, e.g. (5).

$$\left(\frac{y\frac{I}{V}-\frac{z}{W}(AH-BG)}{ab}\right)^2+\left(\frac{y\frac{H}{V}-\frac{z}{W}(CG-AI)}{ac}\right)^2+\left(\frac{y\frac{G}{V}-\frac{z}{W}(BI-CH)}{bc}\right)^2=\left(\frac{x\frac{A}{U}+y\frac{D}{V}+z\frac{G}{W}-A}{a}\right)^2+\left(\frac{x\frac{B}{U}+y\frac{E}{V}+z\frac{H}{W}-B}{b}\right)^2+\left(\frac{x\frac{C}{U}+y\frac{F}{V}+z\frac{I}{W}-C}{c}\right)^2 \quad (8)$$

This is then the equation of the conical surface with apex at $(U;\ 0;\ 0)$ and tangent to the (now rotated) ellipsoid. Its intersection with the "ideal film plane" ($x=0$) is obtained simply by setting $x=0$ in (8). After rearrangement, this yields the ellipse

$$dy^2+ez^2+fyz+gy+hz+i=0 \quad (9)$$

where

$$d=\frac{1}{V^2}\left(\frac{I^2}{a^2b^2}+\frac{H^2}{a^2c^2}+\frac{G^2}{b^2c^2}-\frac{D^2}{a^2}-\frac{E^2}{b^2}-\frac{F^2}{c^2}\right)$$

$$e=\frac{1}{W^2}\left(\frac{(BG-AH)^2}{a^2b^2}+\frac{(AI-CG)^2}{a^2c^2}+\frac{(CH-BI)^2}{b^2c^2}-\frac{G^2}{a^2}-\frac{H^2}{b^2}-\frac{I^2}{c^2}\right)$$

$$f=\frac{2}{VW}\left(\frac{I(BG-AH)}{a^2b^2}+\frac{H(AI-CG)}{a^2c^2}+\frac{G(CH-BI)}{b^2c^2}-\frac{DG}{a^2}-\frac{EH}{b^2}-\frac{FI}{c^2}\right)$$

$$g=\frac{2}{V}\left(\frac{AD}{a^2}+\frac{BE}{b^2}+\frac{CF}{c^2}\right)$$

$$h=\frac{2}{W}\left(\frac{AG}{a^2}+\frac{BH}{b^2}+\frac{CI}{c^2}\right)$$

$$i=-\left(\frac{A^2}{a^2}+\frac{B^2}{b^2}+\frac{C^2}{c^2}\right)$$

Similar reasoning gives the equation for the projection of the ellipsoid onto the xz-plane, which is similar to (9), except that y, A, B and C have been interchanged with x, D, E, and F, respectively, and that the sign of f is negative.

APPENDIX II

Calculation of various parameters of an ellipse of the form

$$dx^2+ey^2+fxv+gx+hy+i=0 \quad (9)$$

(a) Calculation of the main axes of the ellipse. If f is negative, the equation is first multiplied by -1. The Euclidean transformation

$$x=x_1\sqrt{\frac{1-R}{2}}+y_1\sqrt{\frac{1+R}{2}}$$

$$y=-x_1\sqrt{\frac{1+R}{2}}+y_1\sqrt{\frac{1-R}{2}}$$

where

$$R=\frac{d-e}{\sqrt{f^2+(d-e)^2}}$$

rotates the ellipse into the ellipse

$$d_1x_1^2+e_1y_1^2+g_1x_1+h_1y_1+i_1=0$$

where

$$d_1=\frac{1}{2}\left(d(1-R)+e(1+R)-f\sqrt{1-R^2}\right)$$

$$e_1=\frac{1}{2}\left(d(1+R)+e(1-R)+f\sqrt{1-R^2}\right)$$

$$g_1=\frac{1}{\sqrt{2}}\left(g\sqrt{1-R}-h\sqrt{1+R}\right)$$

$$h_1=\frac{1}{\sqrt{2}}\left(g\sqrt{1+R}+h\sqrt{1-R}\right)$$

$$i_1=i$$

Since the x_1y_1 term is zero, the axes of the ellipse are parallel to the coordinate axes. The translation

$$x_1=x_2-\frac{g_1}{2d_1}$$

$$y_1=y_2-\frac{h_1}{2e_1}$$

brings the equation onto the form

$$d_2x_2^2+e_2y_2^2+i_2=0$$

where

$$d_2=d_1$$

$$e_2=e_1$$

$$i_2=i_1-\frac{g_1^2}{4d_1}-\frac{h_1^2}{4e_1}$$

This is the equation of an ellipse, centered at the origin. Its axes are

$$l_1=\sqrt{\frac{-i_2}{d_2}}$$

$$l_2=\sqrt{\frac{-i_2}{e_2}}$$

(*b*) *Calculation of the overall width and height of the ellipse (Fig. 4).* A line $y=k$, parallel to the x-axis, cuts the ellipse at at most two points, whose x-coordinates are given by the equation

$$dx^2+ek^2+fxk+gx+kh+i=0$$

If the line is a tangent to the ellipse, these two points coincide, which happens if the discriminant of the equation is zero, i.e. if

$$\left(\frac{fk+g}{2}\right)^2-d(ek^2+hk+i)=0$$

This is an equation in k with two roots

$$k_{1,2}=\frac{2dh-fg}{f^2-4de}\pm\frac{2\sqrt{d}\sqrt{dh^2-fgh-4dei+f^2i+eg^2}}{f^2-4de}$$

Thus the lines $y=k_1$ and $y=k_2$ are the two horizontal tangents of the ellipse, and the distance between them is obviously

$$l_2=\frac{4\sqrt{d}\sqrt{dh^2+eg^2-4dei-fgh+f^2i}}{|f^2-4de|}$$

For the distance l_1 between the vertical tangents, similar reasoning yields the same formula, with $\sqrt{e}$ substituted for $\sqrt{d}$.

(*c*) *Calculation of the longest chord of the ellipse parallel to the x-axis.* From Fig. 4 it is seen that the longest such chord is the one passing through the centre of the ellipse, i.e.

$$y=\frac{k_1+k_2}{2}=\frac{2dh-fg}{f^2-4de}$$

If this value is inserted into eq. (9), an equation in x results, whose roots

$$x_{1,2}=-\frac{f\frac{2dh-fg}{f^2-4de}+g}{d}\pm\frac{1}{2}\sqrt{\frac{(g^2-4di)(f^2-4de)-(2dh-fg)^2}{d^2(f^2-4de)}}$$

are the x-coordinates of the intersections between the line and the ellipse. Evidently the distance between them is

$$l_3=\frac{1}{|d|}\sqrt{\frac{(g^2-4di)(f^2-4de)-(2dh-fg)^2}{(f^2-4de)}}$$

APPENDIX III

Calculation of the length of the sagittal diameter of the ellipsoid

In the x_0, y_0, z_0-system, the sagittal diameter is a line through the centre of the heart, passing through $(A; B; C)$. Its equation is

$$x_0=At$$

$$y_0=Bt$$

$$z_0=Ct \qquad (10)$$

If these equations are entered into (2), an equation in t results, with the roots

$$t_{1,2}=\pm\frac{1}{\sqrt{\frac{A^2}{a^2}+\frac{B^2}{b^2}+\frac{C^2}{c^2}}}$$

If these values are entered into (10), the coordinates of the endpoints of the diameter are obtained. The distance between these points is

$$\frac{2U}{\sqrt{\frac{A^2}{a^2}+\frac{B^2}{b^2}+\frac{C^2}{c^2}}}$$

ACKNOWLEDGEMENTS

I want to thank Dr Uno Eriksson, Dr Kjell Bergström and Mr Gunnar Ruhn for valuable discussions and criticism.

REFERENCES

1. Bergström, K.: Roentgenological heart volume determination in model experiments. Acta Soc Med Upsal *74:* 49, 1969.
2. Jonsell, S.: A method for the determination of the heart size by teleroentgenography (a heart volume index). Acta Radiol (Stockh.) *20:* 325, 1939.
3. Kahlstorf, A.: Über eine orthodiagraphische Herzvolumenbestimmung. Fortschr Röntgenstr *45:* 123, 1932.
4. Kjellberg, S. R., Lönroth, H. & Rudhe, H.: The effect of various factors on the roentgenological determination of the cardiac volume. Acta Radiol (Stockh.) *35:* 413, 1951.
5. Shilov, G. E.: An Introduction to the Theory of Linear Spaces. Prentice-Hall, Englewood Cliffs, 1961.

Received October 10, 1977

Address for reprints:

Göran Sperber, M.D.
Department of Physiology and Medical Biophysics
Biomedical Centre
Box 572
S-751 23 Uppsala
Sweden